Localization

for affine W-algebras

plan:

1^{st} $1/2$: recall some salient features from rep thry of W-algebras

(Kac, Feigin, Fuchs, Frenkel, Felder, Wakimoto, Arakawa)

2^{nd} $1/2$: describe corresponding localizations

(jt w/ Raskin)

pt 1 : rep theory
of W-algebras

1.1 Virasoro (W-algebra for sl_2)

Witt algebra:

$$\mathbb{C}((t))\partial_t$$

usual
Lie bracket

$$L_n \sim -t^{n+1}\partial_t \,, \quad n \in \mathbb{Z}:$$

$$[L_m, L_n] = (m-n)\, L_{m+n}$$

$$[f(t)\partial_t,\ g(t)\partial_t] = \left(f(t)\,\dot{g}(t) - \dot{f}(t)\,g(t) \right) \partial_t$$

$$0 \to \mathbb{C}c \to \mathrm{Vir} \to \mathrm{Witt} \to 0$$

1. c central $[c, -] = 0$

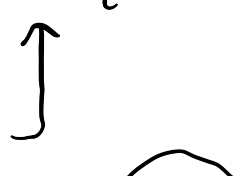
2. $[L_m, L_n] = (m-n) L_{m+n} + \delta_{m,-n} \dfrac{(m-1)m(m+1)}{12} c$

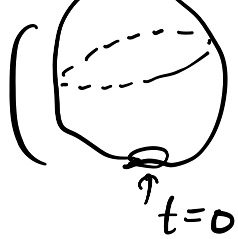
<u>rmk.</u>

<u>Witt</u> $\sim$ Vect ()

$t=0$

$$\begin{matrix} L_1 \\ L_0 \\ L_{-1} \end{matrix}$$



$\underline{\underline{sl_2}} \quad \simeq \quad \text{Vect}\Big($ 

$\Big)$

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classify reps of Vir ?

lemma Any f.d. Vir-module is
smooth

a $\oplus$ of trivial rep.

pf. If M f.d. irrep,

consider action $Vir^{>0} \curvearrowright M$

$\wr$

span $L_1 \; L_2 \; L_3 \; L_4 \; \ldots$

① this prosolvable

② $L_{-n} \quad [L_{-n}, L_n] \quad L_n \quad \sim \quad sl_2$

③ $\exists \; m \in M$ killed by $Vir^{>0}$.

④ consider $Vir^0 \circlearrowright M^{Vir^{>0}}$

$$\begin{array}{cc} Vir^0 & \\ \wr & \\ L_0 & c \\ \shortparallel & \\ -t\partial_t & \end{array}$$

c acts by a scalar $\Gamma \in \mathbb{C}$

L_0 act semisimply

$\Rightarrow$ M gen by a h.w. vector:

$$L_i \cdot m = 0 \qquad i > 0$$

$$L_0 \cdot m = \Delta m \qquad \underline{\Delta} \in \mathbb{C}$$

$$c \cdot m = \Gamma \cdot m \qquad \underline{\Gamma} \in \mathbb{C}$$

for M to be f.d. need in particular that acting by each $\langle L_{-n}, [L_{-n}, L_n], L_n \rangle$ m should be f.d. w/ a uniform bound independent of n.

$$\boxed{[L_n, L_{-n}] = \underline{2n} \underbrace{L_0}_{h'} + \frac{n(n+1)(n-1)}{12} \cdot c}$$

$$[h, L_n] = 2L_n \qquad h = -\frac{h'}{n^2}$$

$$[h', L_n] = -2n^2$$

$h \circlearrowleft m$ h.w. vector by

$$-\frac{2n\Delta}{n^2} + -\frac{n(n+1)(n-1)}{12n^2} \cdot c$$

$$= -\frac{2\Delta}{n} + -\frac{(n+1)(n-1)}{12n} \cdot c$$

as $n \to \infty$ $\quad \approx -\frac{n}{12} c$

$$\implies c = 0$$

but $-\frac{2\Delta}{n} \in \mathbb{Z}^{\geq 0}$

$$-2\Delta \in n\mathbb{Z}^{\geq 0} \quad \forall\, n \gg 0$$

$$\implies \Delta = 0$$

$\implies$ M was trivial module! $\square$

$$\begin{tikzcd} & \mathrm{Vir}^{\geq 0} \arrow[dl] \arrow[dr] & \\ \mathrm{Vir}^{0} & & \mathrm{Vir} \end{tikzcd}$$

$c \quad L_0 \quad L_1 \quad L_2 \; \cdots$

$c \quad L_0$

<u>def</u> For $c, \Delta \in \mathbb{C}$

Verma module

$$M_{c,\Delta} = \mathrm{Ind}_{\mathrm{Vir}^0}^{\mathrm{Vir}} \mathbb{C}_{c,\Delta}$$

??

$\vdots \qquad \vdots \qquad \vdots$

$L_{-1}^2 |c,\Delta\rangle \qquad L_{-2}|c,\Delta\rangle \qquad L_0 \equiv \Delta+2$

$L_{-1}|c,\Delta\rangle \qquad L_0 \equiv \Delta+1$

$|c,\Delta\rangle \qquad L_0 \equiv \Delta$

$$\mathbb{C}[L_{-1}, L_{-2}, \dots] \cdot |c, \Delta\rangle .$$

$$\operatorname{ch} M_{c,\Delta} = \frac{q^{\Delta}}{\prod_{j=1}^{\infty} (1-q^j)}$$

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so for:

$$0 \to K \longrightarrow M_{0,0} \longrightarrow \mathbb{C} \to 0$$

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kernel ~ augmentation ideal of

$$U(Vir^{<0})$$

$$L_{-2}|0,0\rangle$$

$L_{-1}|0,0\rangle$

$|0,0\rangle$

① $Vir^{<0}$ gen by L_{-1} & L_{-2}

$\Rightarrow$ any gen by $L_{-1}|0,0\rangle$

$L_{-2}|0,0\rangle$

② both $L_{-1}|0,0\rangle$ are again

$-\frac{3}{2}L_{-1}^2|0,0\rangle + L_{-2}|0,0\rangle$ h.w. vectors

i.e. for $-\frac{3}{2}L_{-1}^2 + L_{-2}$:

$$L_1 \cdot L_{-2}|0,0\rangle \overset{?}{=} 0$$

$$L_2 \cdot L_{-2}|0,0\rangle \overset{?}{=} 0$$

0

$L_{-2} L_1 |0,0\rangle \quad + \quad [L_1, L_{-2}]|0,0\rangle$

$\parallel$

$L_{-1}|0,0\rangle$ h.w. vector

$L_{-2}|0,0\rangle$ h.w. modulo sub gen by $L_{-1}|0,0\rangle$

+ explicitly lift. $\left(-\frac{3}{2} L_{-1}^2|0\rangle\right)$

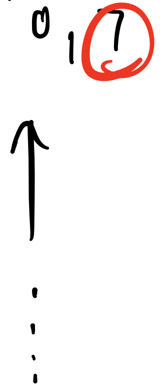
BGG resolution (Feigin-Fuchs, Wallach, Rocha-Caridi)

$$\begin{array}{ccccccc} M_{0,1} & & & & & & \\ \oplus & \longrightarrow & M_{0,0} & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \\ M_{0,2} & & & & & & \\ \uparrow & & & & & & \\ M_{0,5} & & & & & & \\ \oplus & & & & & & \\ M & & & & & & \end{array}$$

$0, 1, 2, 5, 7, \ldots$

pentagonal #s:

χ char:

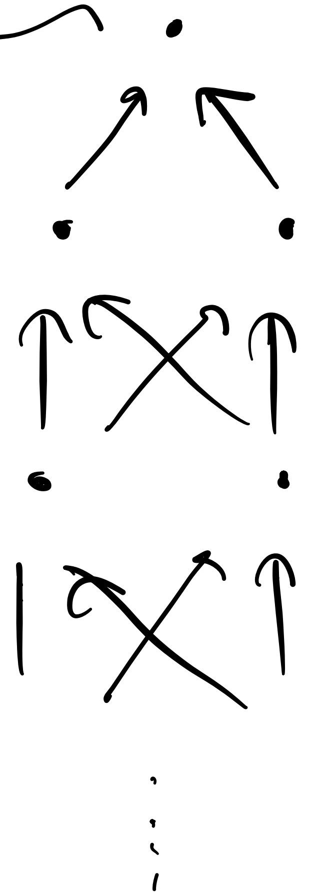


p_n = # part of n

$= p_{n-1} + p_{n-2} - p_{n-5} - p_{n-7}$ ...

(Euler)

$M_{0,0}$



picture of Verma embeddings

$\boxed{c = 0}$

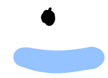
more generally,

Kac Feign Fuchs

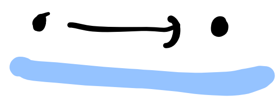
classified all homomorphisms

between Verma modules :

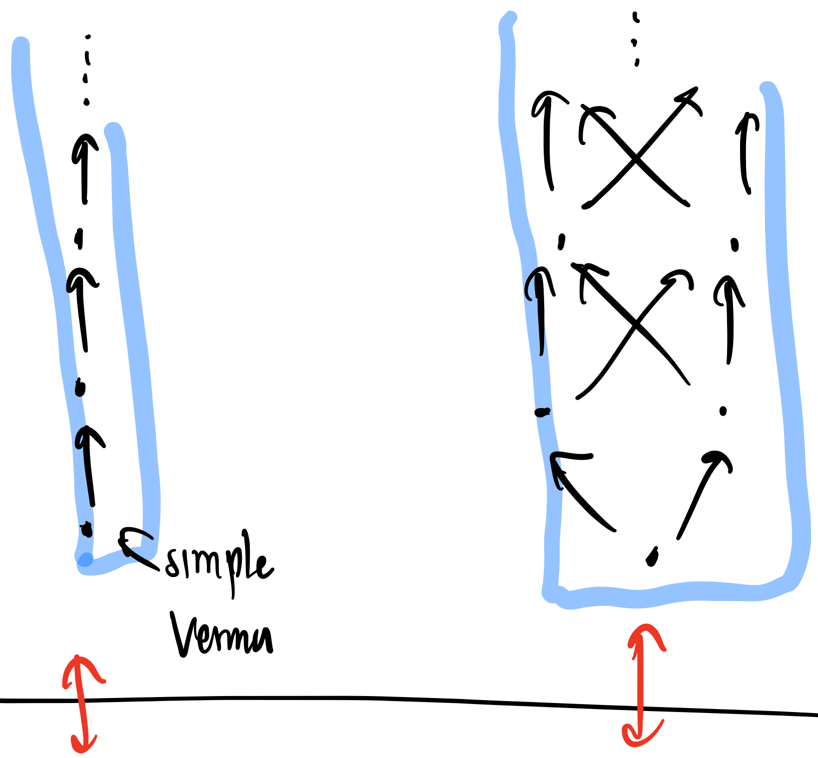
① generic



② subgeneric

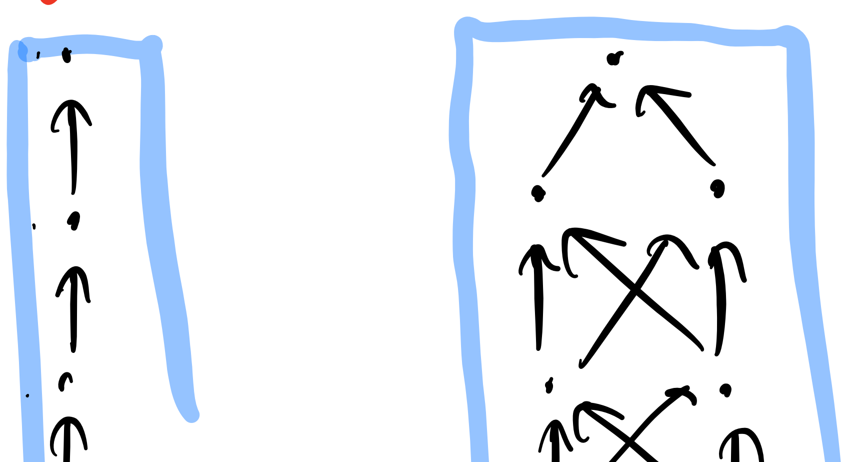


③



$c \in \underline{25} + \mathbb{Q}^{>0}$

④



$c \in \underline{1} - \mathbb{Q}^{>0}$

BRST : $\mathrm{Vir}_{c=26} \longrightarrow \mathrm{Vect}$

duality of Verma embeddings

$\updownarrow$

semi-infinite homological duality:

$\mathfrak{g}$ ss lie algebra:

$\lambda \in \mathfrak{t}^*$

$$\underline{M}_\lambda \longmapsto \underline{M_{-\lambda-2\rho}}$$

flips verma embeddings

$$M \longmapsto R\mathrm{Hom}(M, \mathcal{U}(\mathfrak{g}))[\ ]$$

Vir: (Feigin–Fuchs duality)

$$M_{c,\Delta} \longmapsto M_{26-c,\,1-\Delta}\,.$$

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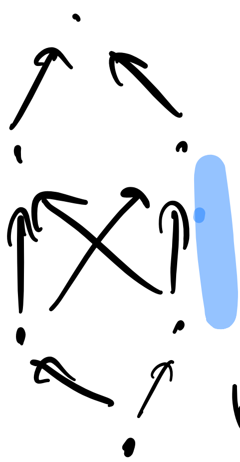
what should:

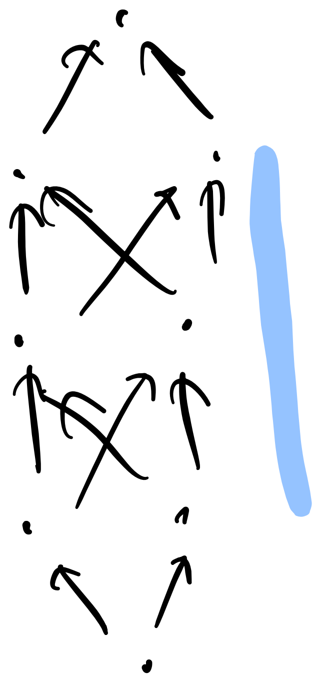
remind you of?

$\mathfrak{sl}_3$:



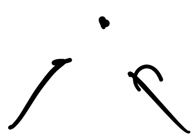
$m_{st}=3$

$\mathfrak{sp}_4$

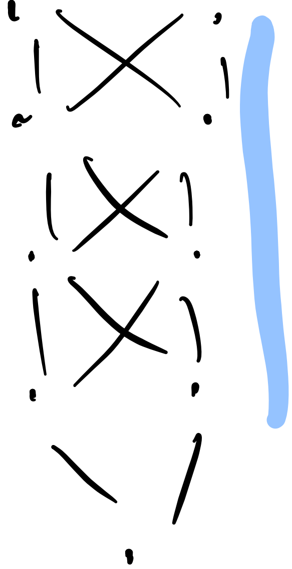


$m_{st}=4$

braid relation in weyl group



$\mathfrak{g}_2$



$m_{st} = 6$

Q: why does Virasoro seem to behave like a infinite dihedral group?

similar Q: for higher rank W-algebras?

A: Virasoro as DS reduction of

$\widehat{\mathfrak{sl}_2}$:

Weyl group $\widehat{\mathfrak{sl}_2}$ is infinite dihedral group.

$\widehat{\mathfrak{sl}_2}$: Verma embeddings :

you see the same pictures.

$$\Psi^+ : \widehat{\mathfrak{sl}_2}\text{-mod} \longrightarrow \mathrm{Vir}\text{-mod}$$

$$\Psi^+ (\mathbb{V}_{\mathfrak{sl}_2, k}) = \mathrm{Vir}_{c(k)}.$$

Virasoro vertex algebra

Feigin–Frenkel (90):

$$\Psi^+ : \mathcal{O}(\widehat{\mathfrak{sl}}_2) \longrightarrow \mathcal{O}(Vir)$$

Vermas $\longmapsto$ Vermas

simples $\longmapsto$ simples or zero

(explicit criterion)

t-exact

$\therefore$ char form on RHS follow from char form on LHS (dihedral group?).

(Frenkel–Kac–Wakimoto, Arakawa):

$$\Psi^{-} : \underline{\mathcal{O}(\hat{\mathfrak{g}})} \longrightarrow \mathcal{O}(\mathcal{W})$$

t-exact

Vermas $\mapsto$ Vermas

simple $\mapsto$ simple or zero

<u>upshot:</u> $\mathcal{O}(\mathcal{W})$ controlled by

affine Weyl group

KL combinatorics.

<u>analogy:</u> Cat $\mathcal{O}$ s.s.

Lie algebra:

T : regular block $\longrightarrow$ <u>singular block</u>

$\uparrow$

translation

T : t-exact

Verma $\mapsto$ Verma

simple $\mapsto$ simple or zero

on k_0 :

$$\mathbb{Z}[W] \longrightarrow \mathbb{Z}[W/W_J]$$

$\uparrow$ parabolic subgroup

(de Vost - van Onel)

for W :

$$\text{block for } \hat{\mathfrak{g}} \xrightarrow{\psi^-} \text{block for } W$$

$$\mathbb{Z}[W^{aff}_\lambda / W_J] \longrightarrow \mathbb{Z}[W_{fin} \backslash W^{aff}_\lambda / W_J]$$

$\uparrow$

sing of block

W_λ^{aff}

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remark: In general FKWA $\Rightarrow$ char formulas for W are given via bi-parabolic affine KL polynomials.

Q: how can we make precise the hunch that not just char forms but block only depends on:

$$W_{\lambda,\mathrm{aff}} \cap W_{\mathrm{fin}} \backslash W_\lambda^{\mathrm{aff}} / W_{\lambda,J}$$

$$\mathbb{Z}\left[W_P \backslash W / W_Q\right]$$ i.e. just Coxeter combinatorics?

1st model: $\mathbb{Z} \underset{\mathbb{Z}[W_P]}{\otimes} \mathbb{Z}[W] \underset{\mathbb{Z}[W_Q]}{\otimes} \mathbb{Z}$

$$\operatorname{Perv}\left(P \backslash G / Q \right)$$

Bruhat decomposition

$$G = \bigsqcup_{w \in W_P \backslash W / W_Q} PwQ$$

this almost is what one wants, but not quite:

$$\begin{tikzcd} B\backslash G/B \arrow[d] \\ P\backslash G/B \end{tikzcd}$$

this does <u>not</u> send simples to simples or zero.

<u>instead:</u> triv char $\to$ sgn char :

want Koszul dual cat of perverse sheaves:

$$\begin{tikzcd} B\backslash G/B \arrow[d] \\ N^-_{,\psi}\backslash G/B \end{tikzcd}$$

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cat we're after:

$$\mathcal{O}(\mathfrak{W})_\lambda \simeq$$

$$\mathrm{Shv}\left(\mathring{I}_{1,\psi'_{\lambda},\mathrm{st}} \backslash^{G_\lambda} / \mathring{I}_{1,\psi_{\lambda^0},\mathrm{st}}\right)$$

$$\wr\!\wr$$

$$\mathrm{Shv}\left(N_{,\psi}\backslash^{L_P}/B\right) \otimes \mathrm{Shv}\left(I\backslash^{G_\lambda}/I\right) \otimes \mathrm{Shv}\left(B\backslash^{L_Q}/N_{,\psi}\right)$$

$$\mathrm{Shv}\left(B\backslash^{L_P}/B\right) \qquad \mathrm{Shv}\left(B\backslash^{L_Q}/B\right)$$

nondeg char for L_P $N \subset L_P$

Int Weyl group $\subset \underline{W_{aff}}$

$$\mathrm{sgn} \otimes \mathbb{Z}[W_\lambda] \otimes \mathrm{sgn}$$

$$\mathbb{Z}[W_\lambda \cap W_{fin}] \qquad \mathbb{Z}[W_\lambda^\circ]$$

stab of $\lambda \in \mathfrak{t}^*$

block $\mathcal{O}(W)$

Int Weyl group $\subset \underline{W_{aff}}$

$$\mathbb{Z}[W_\lambda] \otimes \mathrm{sgn}$$

$$\mathbb{Z}[W_\lambda^\circ]$$

ψ^-

block $\mathcal{O}(\hat{\mathfrak{g}})$

Ψ^+ vs. Ψ^-

$$\mathfrak{n}((t))dt \rightrightarrows \mathbb{C}$$

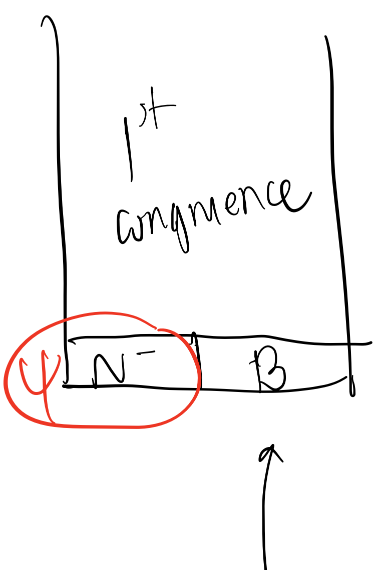
Ψ^+ supp on $\mathfrak{n}\otimes \frac{dt}{t} \xrightarrow{\Psi_{fin}} \mathbb{C}$

$\uparrow$

$e_i \otimes \frac{dt}{t} \mapsto 1$

$i \in I$ simple generators

Ψ^- supp on $\mathfrak{n}\otimes dt$



$\mathfrak{g}[[t]]$

$$\begin{tikzcd} I \arrow[r, "Av"] \arrow[d, equal] & \overset{\circ}{I}^{-}_{,\psi} \arrow[d, equal] \\ ev^{-1}(B) & ev^{-1}(N^{-}) \end{tikzcd}$$

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explain localization theorem (w/ Raskin):

<u>idea:</u> $W\text{-mod} \simeq \hat{\mathfrak{g}}\text{-mod}^{N_F,\psi}$

$$\underline{\underline{\hat{\mathfrak{g}}_\kappa\text{-mod}}} \overset{loc}{\sim} \boxed{D\text{-mod}_\kappa(Fl^{aff}_{G,\lambda})}$$

$$\hat{\mathfrak{g}}_k\text{-mod}^{N_{F,\psi}} \qquad \mathcal{D}\text{-mod}_k\left(N_{F,\psi}\backslash \mathrm{Fl}^{\mathrm{aff}}_{G,d}\right).$$

<u>thm</u> (D. -Raskin)

$k < 0$: t-exact functor:

$$\begin{array}{ccc} \mathrm{Shv}\left(\mathring{I},\psi\backslash \mathrm{Fl}^{\mathrm{aff}}_G\right) & & \\ \wr\!\!\mid & \Gamma & \\ \mathrm{Shv}_k\left(N_{F,\psi}\backslash \mathrm{Fl}^{\mathrm{aff}}_G\right) & \hookrightarrow & \mathcal{W}\text{-mod}_k \end{array}$$

(for ~~almost~~ almost all k, d) embedding of a $\oplus$ of blocks

$k \geq 0$: $\quad \mathrm{Bun}_G(\mathbb{P}^1)$ + $\mathcal{B}$ reduction at $\infty \in \mathbb{P}^1$

$$\mathrm{Shv}_k(\mathring{I}_{,\varphi} \backslash \mathrm{Fl}) \xhookrightarrow{\Gamma} W_k\text{-mod}$$

t-exact embedding of a $\oplus$ of blocks

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rmk similar pictures + formulas for every nilpotent orbit.
